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STRUCTURE OF ELASTIC DISCONTINUITIES ON SHOCK PROFILES IN VISCOELASTIC MEDIA

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Godunov and Kozin [1] have investigated the structure of shock waves in a viscoelastic medium characterized by a tangential-stress relaxation time τ and a special type of elastic energy equation for the medium. The present author [2] has formulated inequality-type constraints for the elastic energy, which are sufficient conditions for the shock-structure problem with a specified particle velocity and a structure of the form in [1] to have a unique solution. The indicated shock structure has the singular attribute that for wave velocities greater than the velocity of sound the wave profile suffers a discontinuity, which corresponds to a jump-type variation in the elastic constants of the medium. Mathematical relations at the discontinuity have been derived in [1] from heuristic considerations, and to justify the selected type of relations it is necessary to formulate a small dissipation mechanism that would produce shock "smearing" by an amount of the order of the characteristic dissipation scale. The dissipation mechanism introduced in the present study is viscous friction with a small viscosity coefficient μ; it is shown that under the constraints imposed in [2] on the equations of state positive μ corresponds to a unique continuous solution of the shock-structure problem. It is also shown that as μ tends to zero, the solution of the "shock-smearing" problem tends to the solution of the problem in [1, 2]. Inasmuch as these tendencies are nonuniform in the case of supersonic shock types, the limit solution contains a mathematical discontinuity satisfying the shock relations [1].

§1. Conditions on the Elastic Energy Equation

Let us consider a homogeneous isotropic medium with internal energy density per unit mass given by the equation

$$E = E(\alpha,\ \beta,\ \gamma,\ S), \tag{1.1}$$

in which E is a symmetric function of the parameters α, β, γ, which represent the logarithms of the relative elongations k_1, k_2, k_3 along the principal strain axes, and S is the entropy density per unit mass. Following [2], we assume that expression (1.1) satisfies the inequalities

$$T = \frac{\partial E}{\partial S} > 0, \quad r = \left(\frac{\partial E}{\partial \alpha} - \frac{\partial E}{\partial \beta}\right)\Big/(\alpha - \beta) > 0; \tag{1.2}$$

$$c^2 = \frac{\partial^2 E}{\partial \alpha^2} - \frac{\partial E}{\partial \alpha} > 0, \quad l = \frac{\partial^2 E}{\partial \alpha \partial S} < 0; \tag{1.3}$$

$$h = \frac{\partial^2 E}{\partial \alpha \partial \beta} - \frac{\partial^2 E}{\partial \alpha^2} - \frac{\partial^2 E}{\partial \alpha \partial S}\left(\frac{\partial E}{\partial \beta} - \frac{\partial E}{\partial \alpha}\right)\Big/\frac{\partial E}{\partial S} < 0; \tag{1.4}$$

$$q = \frac{\partial^3 E}{\partial \alpha^3} - 3\frac{\partial^2 E}{\partial \alpha^2} + 2\frac{\partial E}{\partial \alpha} < 0, \quad a^2 = c^2 + \frac{2}{3}\left(\frac{\partial^2 E}{\partial \alpha \partial \beta} - \frac{\partial^2 E}{\partial \alpha^2}\right) > 0, \tag{1.5}$$

as well as the inequalities obtained from (1.2)-(1.5) by cyclic substitution of indices.

§2. One-Dimensional Equations

The system of differential equations describing the motion of a viscoelastic medium parallel to the x axis in space (x, y, z) has the form

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$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u}{\partial x}=0,\quad \frac{\partial\rho u}{\partial t}+\frac{\partial\left(\rho u^2+p-\mu\frac{\partial u}{\partial x}\right)}{\partial x}=0,$$

$$\frac{\partial\rho\left(E+\frac{u^2}{2}\right)}{\partial t}+\frac{\partial\left[\rho u\left(E+\frac{u^2}{2}\right)+pu-\mu u\frac{\partial u}{\partial x}\right]}{\partial x}=0, \tag{2.1}$$

$$\frac{\partial\beta}{\partial t}+u\frac{\partial\beta}{\partial x}=-\frac{\beta-\frac{\alpha+\beta+\gamma}{3}}{\tau},$$

where x and t are the spatial coordinates and time, u is the velocity in the x direction, and $\rho = \rho^0 e^{-\alpha-\beta-\gamma}$ is the density. By the isotropicity of the medium we have $\beta=\gamma$. The quantity ρ^0 is the density of the medium under standard conditions. The principal stresses $\sigma_x=-p$, σ_y, σ_z along the x, y, z, axes are related to the strains by the expressions

$$\sigma_x=\rho\partial E/\partial\alpha,\quad \sigma_y=\rho\partial E/\partial\beta,\quad \sigma_z=\rho\partial E/\partial\gamma.$$

Equations (2.1) differ from the system of equations in [1, 2] by the viscous friction terms with small $\mu>0$, which introduce additional energy dissipation in the system.

We next consider the shock structure, i.e., the existence and uniqueness of solutions of the system (2.1) in the form

$$\alpha=\alpha(x-Ut),\quad \beta=\beta(x-Ut)=\gamma(x-Ut),\quad S=S(x-Ut), \tag{2.2}$$

such that the solutions tend to constants as $|x|\to\infty$. In a shock frame moving with velocity U relative to the laboratory frame, the system of equations (2.1) can be transformed as follows for solutions of the form (2.2):

$$\rho u=\rho_0u_0=w,$$

$$E-E_0+(v-v_0)\left(p_0-\frac{w^2}{2}(v-v_0)\right)=H(p,v)=0, \tag{2.3}$$

$$\mu\frac{dv}{d\xi}=p-p_0+w^2(v-v_0)=G(p,v),\quad 3\frac{v\tau}{\mu}\frac{d\beta}{d\xi}=\alpha-\beta,$$

where the counterpart in the second equation of the dissipation term in (2.1) is eliminated by means of the third equation of (2.3). Here the constants u_0, p_0, v_0, and E_0 specify the state of the substance as $\xi\to-\infty$ ("preshock" state) and w is the particle velocity. The parameter $\xi=(x-Ut)/\mu w$ and $v=1/\rho$. The equilibrium points of the system (2.3) are given by the equations $G(p, v)=H(p, v)=0$, $\alpha-\beta=0$ and correspond to the beginning and end of the shock wave. As shown in [2], conditions (1.2)-(1.5) are sufficient in order for the system $G(p, v)=H(p, v)=0$, $\alpha-\beta=0$ to have two and only two solutions: 1) v_0, p_0, or "beginning of shock"; 2) v_1, p_1, or "end of shock." To solve the problem of the shock-profile structure it is sufficient to indicate a unique solution (2.3) passing through the equilibrium points corresponding to the beginning and end of the shock wave.

§ 3. Structure of Equilibrium Points

To analyze the type of equilibrium positions of the system (2.3) we write the appropriate equations in variations. Putting $\delta v=v-v_i$, $\delta\alpha=\alpha-\alpha_i$, $\delta S=S-S_i$, $\delta\beta=\beta-\beta_i$, we obtain

$$\delta H=T\delta S-vG\delta\alpha-2v(G+\rho(E_\alpha-E_\beta))\delta\beta;$$

$$\delta G=\rho^2(w^2v^2-c^2)\delta v-2\rho g\delta\beta-\rho E_{\alpha s}\delta S,$$

where $g=E_{\alpha\beta}-E_{\alpha\alpha}$. From this result we deduce (since $\delta S=0$ at the equilibrium positions) the equations in variations

$$\begin{cases}\dfrac{d(\delta v)}{d\xi}=\rho^2(v^2w^2-c^2)\delta v-2\rho g\delta\beta,\\ 3v\tau^*\dfrac{d(\delta\beta)}{d\xi}=v\delta v-3\delta\beta,\quad \tau^*=\tau/\mu.\end{cases} \tag{3.1}$$

The values of the coefficients (3.1) are computed at the equilibrium points of the system (2.3). The characteristic determinant of the system (3.1) has the form

$$3v\tau^*\lambda^2+3(1-\tau^*(w^2v^2-c^2))\lambda+2g\rho-3\rho^2(v^2w^2-c^2)=0. \tag{3.2}$$

The eigenvalues of Eq. (3.2) are real and distinct, because the discriminant

$$\Delta=9(1+\tau^*(v^2w^2-c^2))^2-24\tau^*g>0$$

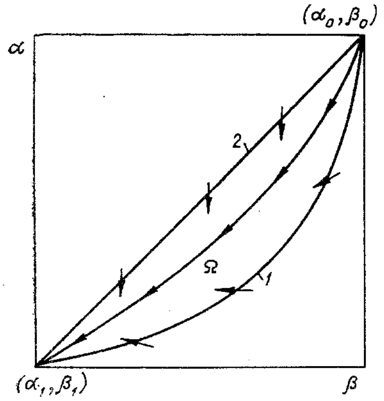

Fig. 1

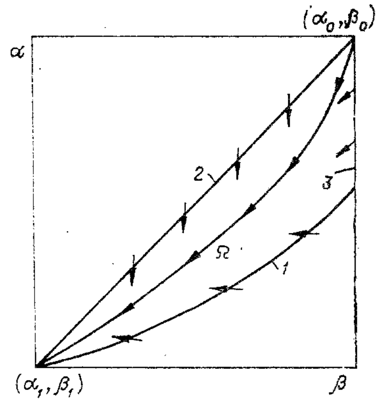

Fig. 2

and $g = h|_{\alpha=\beta} < 0$ on the basis of (1.4). The product and sum of λ_1 and λ_2 are given by the relations

$$\lambda_1\lambda_2 = (\rho^2/\tau^*)(a^2 - v^2w^2); \quad \lambda_1 + \lambda_2 = \rho(v^2w^2 - c^2 - 1/\tau^*). \tag{3.3}$$

The eigenvalues λ_1 and λ_2 correspond to the eigenvectors

$$\delta S_i = 0, \quad [\rho^2(v^2w^2 - c^2) - \lambda_i]\delta v_i - 2\rho g\delta\beta_i = 0, \quad i = 1,\ 2,$$

which correspond in turn to the separatrices of the equilibrium positions. The small viscosity μ corresponds to large values of the quantity $\tau^* = \tau/\mu$ and to small values of $\mu^* = \mu/\tau$. The asymptotic representation of the eigenvalues and corresponding eigenvectors for small μ^* has the form

$$\begin{cases} \lambda_1 \infty \Lambda_1 = \dfrac{w^2v^2 - c^2}{v} + A\mu^*, \\ \lambda_2 \infty \Lambda_2 = \dfrac{\rho\,(a^2 - v^2w^2)}{v^2w^2 - c^2}\,\mu^*; \end{cases}$$
$$\delta\beta_1 \infty \frac{A\mu^*}{2\rho g}\,\delta v_1, \qquad \delta\beta_2 \infty \frac{\rho^2\,(v^2w^2 - c^2) - \Lambda_2\mu^*}{2\rho g}\,\delta v_2, \tag{3.4}$$

where

$$A = \rho\,\frac{a^2 + c^2 - 2v^2w^2}{c^2 - v^2w^2}. \tag{3.5}$$

The foregoing asymptotic expression enables us to investigate the structure of the equilibrium positions.

1. Initial Point. For $\alpha = \beta$ the quantity a^2 represents the propagation velocity of small volume waves. We consider only shock propagation velocities for which $w^2v_0^2 > a_0^2$, i.e., shocks moving with velocities greater than the preshock volume velocity of sound. It follows from (3.3) in this case that $\lambda_1^{(0)}\lambda_2^{(0)} < 0$, i.e., the initial point is a saddle point; the superscript (0) denotes the initial point of the shock and the superscript (1) the end point of the shock. The separatrices of the saddle point are given by (3.5) for small μ. Variation of the parameter ξ in the direction from $-\infty$ to $+\infty$ corresponds in the phase plane of the system (2.3) to motion along the curve describing the shock profile, from the beginning of the latter to the end. Consequently, only a "departing" separatrix of the initial point can correspond to a real shock. We infer from (3.4) that the required separatrix in the subsonic case ($w^2v_0^2 - c_0^2 < 0$) is an integral curve with slope $\delta\beta = \dfrac{\rho_0^2\left(v_0^2w^2 - c_0^2\right) - \Lambda_2\mu^*}{2\rho_0 g_0}\,\delta v$ and in the supersonic case ($w^2v_0^2 - c_0^2 > 0$) is an integral curve with slope $\delta\beta = \mu^*\dfrac{a_0^2 + c_0^2 - 2v_0^2w^2}{2g_0\left(c_0^2 - w^2v_0^2\right)}\,\delta v$. The rate of motion along the curves near the equilibrium position in this case is characterized by the factor $e^{\Lambda_2\mu^*\xi}$ in the subsonic case and $e^{\Lambda_1\xi}$ in the supersonic.

2. End Point. The equations $G(p, v) = H(p, v) = 0$, $\alpha = \beta$ describing the equilibrium positions in the phase plane of the system (2.3) represent the customary shock relations for the gasdynamic equations, because the condition $\alpha = \beta$ implies that the stresses σ_x, σ_y, σ_z before and after the shock are equal. It has been shown [2] that conditions (1.2)-(1.5) are sufficient in order for the condition $w^2v_1^2 - a_1^2 < 0$ to be satisfied at the end point of the shock, i.e., for the postshock regime to be subsonic. This means that $\lambda_1^{(1)} < 0$ and $\lambda_2^{(1)} < 0$ at the end point, i.e., the singularity represents an accumulation point, and so "arrival" at the end point takes place along

all integral curves as $\xi \to +\infty$. Here the separatrix of the accumulation point (at which is tangent an infinite set of integral curves) is the curve corresponding to the value $\lambda_2^{(1)} \sim [\rho_1(a_1^2 - v_1^2 w^2)(w^2 v_1^2 - c_1^2)]\mu^*$ and having the slope $\delta\beta = \{[\rho_1^2(w^2 v_1^2 - c_1^2) - \Lambda_2 \mu^*]/2\rho_1 g_1\}\delta v$.

§ 4. Existence of a Solution

To demonstrate the existence of an integral curve connecting the equilibrium points of the system (2.3) it is necessary to construct in the phase plane of the system [which can be the (α, β) plane] a domain Ω such that both equilibrium points lie on its boundary; integral curves of the system (2.3) can only arrive at it; and, finally, it contains one of the branches of the separatrix of the initial point $\alpha_0 = \beta_0$, corresponding to departure from the saddle point.

The domain Ω is constructed differently for the subsonic $(w^2 v_0^2 - c_0^2 < 0)$ and supersonic $(w^2 v_0^2 - c_0^2 > 0)$ cases. For its construction we examine the curve given by the equation $H(p, v) = G(p, v) = 0$.

Following [1, 2], we refer to this construction as the curve of possible states. It is shown in [2] that when conditions (1.2)-(1.5) are satisfied, the curve of possible states has the form $\alpha = \alpha(v)$, $\beta = \beta(v)$ in the (α, β) plane for $v_1 \le v \le v_0$ in the subsonic case and for $v_1 \le v \le v_2$ in the supersonic case. Here the inequality $v_2 < v_0$ corresponds to the point of intersection (α_2, β_0) of the curve of possible states with the line $\beta = \beta_0$ in the (α, β) plane (Figs. 1 and 2).

For $w^2 v_0^2 - c_0^2 < 0$ the domain Ω is bounded by the curve of possible states and the line $\alpha = \beta$ (curves 1 and 2 in Fig. 1), and for $w^2 v_0^2 - c_0^2 > 0$ by the curve of possible states and the lines $\alpha = \beta$ and $\beta = \beta_0$ (curves 1-3 in Fig. 2). As shown in [2], the relative positions of the curves bounding Ω are as given in Figs. 1 and 2. We now show that the integral curves of the system (2.3) can only enter Ω. The curve of possible states is represented in the regions in question by a function of the parameter v, and so, since $v = v^0 e^{\alpha + 2\beta}$, it is transversal to the field of lines $\alpha + 2\beta = \text{const}$ in those regions. Moreover, in the region where the curve of possible states is the boundary of Ω, it is situated in the half-plane $\beta > \alpha$, whereupon it follows from (2.3) that $d\beta/d\xi > 0$ at curves 1 in Figs. 1 and 2. Hence, as ξ increases, the integral curves can intersect curves 1 in Figs. 1 and 2 only on entering Ω. At curves 2 (Figs. 1 and 2) we have $\beta = \alpha$ and $G(p, v) < 0$ if $H(p, v) = 0$. From this result and from (2.3) we infer that $d\beta/d\xi = 0$ and $dv/d\xi = d\alpha/d\xi < 0$ at curves 2; i.e., the integral curves can only enter Ω via the boundary $\alpha = \beta$ of Ω. For supersonic shock velocities the boundary of Ω includes the line $\beta = \beta_0$ (curve 3 in Fig. 2). Inasmuch as Ω is situated in the plane $\beta > \alpha$, we have $d\beta/d\xi < 0$ along $\beta = \beta_0$, implying that the solutions of (2.3) enter Ω via this boundary as well.

For an integral curve passing through the singular point (β_0, v_0) to enter the domain Ω it is necessary that it have, at (β_0, v_0) a slope $\delta\beta/\delta v$ satisfying the inequalities

$$\begin{cases} \rho_0/3 \leqslant \left(\dfrac{\delta\beta}{\delta v}\right)_0 \leqslant 0 \quad \text{for} \quad w^2 v_0^2 - c_0^2 > 0, \\ \rho_0/3 \leqslant \left(\dfrac{\delta\beta}{\delta v}\right)_0 \leqslant \rho_0 \dfrac{v_0^2 w^2 - c_0^2}{2h_0} \quad \text{for} \quad w^2 v_0^2 - c_0^2 < 0. \end{cases}$$

As the expansions (3.4) and (3.5) imply, inequalities of this type are satisfied for $w^2 v_0^2 - c_0^2 > 0$ by the saddle-point separatrix corresponding to the eigenvalue $\lambda_1^{(0)}$, and for $w^2 v_0^2 - c_0^2 < 0$ by the separatrix corresponding to the eigenvalue $\lambda_2^{(0)}$. Thus, for small values of μ^*

$$\rho_0/3 < \left(\frac{\delta\beta}{\delta v}\right)_1^{(0)} = \frac{a_0^2 + c_0^2 - 2v_0^2 w^2}{c_0^2 - v_0^2 w^2}\mu^* < 0, \quad \text{if} \quad v_0^2 w^2 - c_0^2 > 0;$$

$$\rho_0/3 < \left(\frac{\delta\beta}{\delta v}\right)_2^{(0)} = \frac{\rho_0\left(v_0^2 w^2 - c_0^2\right) - \dfrac{\rho_0\left(a_0^2 - w^2 v_0^2\right)}{v_0^2 w^2 - c_0^2}\mu^*}{2h_0} \leqslant \rho_0 \frac{v_0^2 w^2 - c_0^2}{2h_0},$$

$$\text{if} \quad a_0^2 < v_0^2 w^2 < c_0^2.$$

Consequently, the required solution is that separatrix of the saddle point (β_0, v_0) which, starting at the initial point, enters the domain Ω (see Figs. 1 and 2) and by the foregoing considerations cannot depart from Ω. Inasmuch as Ω does not contain any singular points, the separatrix of the saddle point must terminate at (β_1, v_1).

§ 5. Asymptotic Behavior of the Solution as $\mu \to 0$

It has been shown [2] that conditions (1.1)-(1.5) embody a unique (possibly discontinuous) solution of the boundary-value problem

$$\begin{cases} H = E - E_0 + (v - v_0)\left(p_0 - \frac{w^2}{2}(v - w_0)\right) = 0, \\ G = p - p_0 + w^2(v - v_0) = 0,\ 3\tau v\frac{d\beta}{dx} = \alpha - \beta, \end{cases} \tag{5.1}$$

passing through the points (v_0, p_0) and (v_1, p_1) as $x \to \pm\infty$. The system (5.1) is obtained from (2.3) by passing to the limit $\mu \to 0$, in which case (2.3) is a singularly perturbed system for (5.1). We show that the solution of (5.1) is the limit, in an appropriate sense, for the solution of (2.3) constructed in the preceding sections. The following upper bound is obtained precisely as $\mu \to 0$ for any $\varepsilon > 0$:

$$\max_{\beta_1 \leqslant \beta \leqslant \beta_0 - \varepsilon} |v(\beta, \mu) - v(\beta, 0)| \leqslant e^{-(A^2/\mu)\varepsilon} |v(\beta_0, \mu) - v(\beta_0, 0)|.$$

We consider the difference $\Delta v(\beta, \mu) = v(\beta, \mu) - v(\beta, 0)$ between the solutions of the boundary-value problems (2.3) and (5.1), which satisfies the linear differential equation

$$\mu\frac{d\Delta v}{d\beta} = \left(\frac{\partial F(v, \beta)}{\partial v}\right)_{v=v(\beta,0)} \Delta v,$$

in which $F(v, \beta)$ is given in the form of an implicit function:

$$F(v, \beta) = \frac{p - p_0 + w^2(v - v_0)}{\alpha - \beta} 3v\tau,\ H = 0,$$

and its derivative, evaluated at the solution of the unperturbed problem $v = v(\beta, 0)$, has the form

$$\left(\frac{\partial F}{\partial v}\right)_{v=v(\beta,0)} = \left[\frac{vT(w^2 - \rho^2c^2) - 2E_{\alpha S}w^2(v_0 - v)}{Tv - E_{\alpha S}(v_0 - v)} \frac{3v\tau}{\alpha - \beta}\right]_{v=v(\beta,0)}.$$

As shown in [2], in the interval $\beta_1 \le \beta \le \beta_0$ the solution $v = v(\beta, 0)$ obeys the inequalities

$$w^2 - \rho^2c^2 > 0,\ v_0 - v > 0,\ \alpha - \beta > 0,$$

and, since $E_{\alpha S} < 0$ [cf. (1.3)], $(\partial F)/(\partial v)_{v=v(\beta,0)} > 0$ at (β_1, B_0). Denoting $A^2 = \min_{\beta_1 \leqslant \beta \leqslant \beta_0} \left(\frac{\partial F}{\partial v}\right)_{v=v(\beta,0)}$, we obtain $d\Delta v/d\beta \ge (A^2/\mu)\Delta v$. Integrating the inequality between the limits (β, β_0) with regard for the fact that $d\beta < 0$, we obtain

$$|\Delta v| \leqslant |\Delta v|_{\beta=\beta_0} e^{(A^2/\mu)(\beta - \beta_0)}$$

or $\max_{\beta_1 \leqslant \beta \leqslant \beta_0 - \varepsilon} |\Delta v| \leqslant |\Delta v|_{\beta=\beta_0} e^{-A^2\varepsilon/\mu}$ for any $\varepsilon > 0$.

The foregoing estimates show that the introduction of the energy dissipation mechanism with small (of order μ) internal friction has the effect of smearing elastic (i.e., characterized by the condition β = const) discontinuities on the shock profiles by an amount of order μ, thereby justifying the relations adopted in [1] at those discontinuities.

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